

A Simple Path Integration For the Time Dependent Oscillator

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Abstract

Feynman propagator is calculated for the time dependent harmonic oscillator by converting the problem into a free particle motion

I. Introduction

One way of treating the time dependent oscillator described by the Hamiltonian

$$H = \frac{p^2}{2\mu} + \frac{\mu}{2}\omega^2(t)q^2 \quad (1)$$

is employment of the time-dependent point canonical transformations of type [1]

$$\begin{aligned} x &= f(t)Q \\ p &= \frac{1}{f(t)}P \end{aligned} \quad (2)$$

generated by

$$F_2(q, P, t) = \frac{qP}{f(t)}.$$

The above transformation has been used to obtain the Green function by choosing $f(t)$ to map the problem into the usual oscillator with constant frequency ω_0^2 .

Another method of approaching the problem is the making use of the invariants. The Hamiltonian (1) is known to possess a time-dependent invariant given by [2]

$$I = \frac{1}{2}(\dot{q}p - q\dot{p})^2 + \frac{\omega_0^2}{2}\left(\frac{q}{p}\right)^2, \quad \omega_0^2 = \text{constant} \quad (3)$$

The above constant ω_0^2 and the constant frequency of the usual oscillator into which the original problem mapped by (2) are in fact the same. The treatment which uses the invariant approach usually find the wave functions as well as Feynman propagators again by mapping the problem into the constant frequency ones [3].

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In this note we will show that a more simpler choice for the function $f(t)$ of (2), maps the path integration for the time-dependent original problem into the free motion path integral in Q, P - phase space. In other words the choice we employed for $f(t)$, corresponds to the employment of time-dependent invariant (3) with $\omega_0 = 0$. In fact such a choice for ω_0 has been introduced for writing a general expression for the wave function corresponding to (1) which is of the free wave function form [4]. However it has, to our knowledge not been adopted in path integrations.

In the next section we briefly summarize the employment of the transformations (2) in the path integration for the problem, which is essentially same as the one studied in [1]. We then make our choice for $f(t)$, and write the general expression for the Kernel.

In the last section we simply present the results for some interesting forms of the frequency $\omega(t)$.

II. Path Integration with Time-Dependent Point Canonical Transformations

The probability amplitude for the particle to move from the space-time point q_a, t_a to q_b, t_b under the influence of the time dependent oscillator potential is

$$K(q_a, t_a; q_b, t_b) = \int Dq Dp e^{i \int_{t_a}^{t_b} dt (p\dot{q} - \frac{p^2}{2\mu} - \frac{\mu}{2}\omega^2(t)q^2)} \quad (4)$$

which in its explicit time-sliced form expressed as

$$K(q_a, t_a; q_b, t_b) = \lim_{n \rightarrow \infty} \left(\prod_{j=1}^n \int_{-\infty}^{\infty} dq_j \right) \left(\prod_{j=1}^{n+1} \int_{-\infty}^{\infty} \frac{dp_j}{2\pi} \right) \prod_{j=1}^{n+1} e^{i[p_j(q_j - q_{j-1}) - \frac{\epsilon p_j^2}{2\mu} - \frac{\epsilon\mu}{2}\omega^2(t_j)q_j^2]} \quad (5)$$

with

$$t_j = j\epsilon, \quad (t_b - t_a) = (n+1)\epsilon \quad (6)$$

and

$$q_a, t_a = q_0, t_0; \quad q_b, t_b = q_{n+1}, t_{n+1}. \quad (7)$$

Point canonical transformations (2) transforms Hamiltonian and the Action as

$$H_Q = H + \frac{\partial F_2}{\partial F_2} = \frac{P^2}{2\mu f^2(t)} + \frac{\mu}{2}\omega^2(t)f^2(t)Q^2 - \frac{\dot{f}(t)}{f(t)}QP \quad (8)$$

and

$$\int_{t_a}^{t_b} dt (p\dot{q} - H) = \int_{t_a}^{t_b} dt (P\dot{Q} - H_Q). \quad (9)$$

The path integral measure of (5) transforms as

$$\frac{dp_{n+1}}{2\pi} \prod_{j=1}^n \int_{-\infty}^{\infty} \frac{dq_j dp_j}{2\pi} = \frac{dP_{n+1}}{2\pi f(t_{n+1})} \prod_{j=1}^n \int_{-\infty}^{\infty} \frac{dQ_j dP_j}{2\pi}. \quad (10)$$

The factor $f^{-1}(t_{n+1}) = f^{-1}(t_b)$ can be symmetrized as

$$\frac{1}{f(t_b)} = \frac{1}{\sqrt{f(t_a)f(t_b)}} e^{-i\frac{1}{2} \ln \frac{f(t_b)}{f(t_a)}}. \quad (11)$$

Note that we could have arranged the time-slicing from $j = 0$ to $j = n$; in which case we would have an extra momentum integration over dp_0 instead of dp_{n+1} in (5). The symmetrization procedure would lead the same expression as (10) with the sign in the exponent is reversed. Taking the average of these two time-slicing recipes removes the exponential.

The path integration (5) is, after the employments of transformations given by (7) and (8) and the symmetrization (10) becomes (after making a translation $P \rightarrow P + \dot{f}(t)f(t)\mu$)

$$K(q_a, t_a; q_b, t_b) = \frac{1}{\sqrt{f(t_a)f(t_b)}} e^{\frac{i\mu}{2}(\dot{f}(t_b)f(t_b)Q_b^2 - \dot{f}(t_a)f(t_a)Q_a^2)} \tilde{K}(Q_a, t_a; Q_b, t_b) \quad (12)$$

where

$$\tilde{K}(Q_a, t_a; Q_b, t_b) = \int DQDP e^{i \int_{t_a}^{t_b} dt (P\dot{Q} - \frac{P^2}{2\mu f^2(t)} - \frac{\mu}{2}(\omega^2(t)f^2(t) + \dot{f}(t)f(t))Q^2)} \quad (13)$$

Up to this point we have summarized the known procedure [1]. Now we make the choice for $f(t)$ to satisfy simple equation

$$\ddot{f} + \omega^2(t)f = 0 \quad (14)$$

which is same as the one previously employed in studying the invariant (3) to write a general form for the wave function [4]. With (14) the kernel (13) takes the free propagator form and this admits the simple solution:

$$\tilde{K}(Q_a, t_a; Q_b, t_b) = \sqrt{\frac{\mu}{2i\pi W}} e^{i\frac{\mu}{2W}(Q_b - Q_a)^2} \quad (15)$$

where

$$W = \int_{t_a}^{t_b} \frac{dt}{f^2(t)}. \quad (16)$$

Inserting (14) into (11) we arrive at the final form for the original Kernel:

$$K(q_a, t_a; q_b, t_b) = \sqrt{\frac{\mu}{2i\pi f(t_a)f(t_b)W}} e^{i\frac{\mu}{2}(\frac{\dot{f}(t_b)}{f(t_b)}q_b^2 - \frac{\dot{f}(t_a)}{f(t_a)}q_a^2)} e^{i\frac{\mu}{2W}(\frac{q_b}{f(t_b)} - \frac{q_a}{f(t_a)})^2} \quad (17)$$

It is strait forward to verify that the above expression is indeed the correct Green function for the Hamiltonian (1).

III. Examples and Conclusion

For a given frequency all one has to do is to solve the Riccati equation (14) and then insert the solution into the "free" Kernel in (17) Some examples are the following

(i) $\omega(t) = \omega_0 = \text{constant}$,

$$f(t) = \cos \omega_0 t$$

(ii) An exponentially decaying frequency: $\omega^2(t) = \omega_0^2 e^{-\alpha t}$; $\omega_0, \alpha = \text{constants}$.

$$f(t) = J_0\left(\frac{2\omega_0}{\alpha} e^{-\alpha t/2}\right)$$

(iii) $\omega^2(t) = (\omega_0 \alpha^\beta)^2 t^\beta$; ω_0, α, β are arbitrary constants (i.e., not necessarily integer power).

$$f(t) = \sqrt{t/\omega_0 \alpha^\beta} J_{\frac{1}{\beta+2}}\left(\frac{2\omega_0 \alpha^\beta}{\beta+2} t^{\frac{\beta+2}{2}}\right)$$

(iv) A δ - function pulse frequency given by

$$\omega^2(t) = \omega_0^2 \delta(t - t_0) + \omega_0^4 \theta(t - t_0); \quad \omega_0, t_0 = \text{constants}$$

the function $f(t)$ is

$$f(t) = e^{\omega_0 |t - t_0|}.$$

(v) A broader time dependence around t_o may be given by

$$\omega^2(t) = \frac{\alpha^2}{\cosh^2 \beta(t - t_0)}; \quad \alpha, \beta = \text{constants}$$

is solved for

$$f(t) = P_{i\sqrt{\alpha^2 - 1/4} - 1/2}(\tanh \beta(t - t_0))$$

where P is a Legendre function.

Acknowledgments The authors H. Ahmedov and I. H. Duru thanks the Turkish Academy of Sciences (TUBA) for its support.

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